CHARACTERISATIONS OF CROSSED PRODUCTS BY PARTIAL ACTIONS

JOHN QUIGG AND IAIN RAEBURN

Abstract. Partial actions of discrete groups on C^*-algebras and the associated crossed products have been studied by Exel and McClanahan. We characterise these crossed products in terms of the spectral subspaces of the dual coaction, generalising and simplifying a theorem of Exel for single partial automorphisms. We then use this characterisation to identify the Cuntz algebras and the Toeplitz algebras of Nica as crossed products by partial actions.

Introduction

Exel has recently introduced and studied partial automorphisms of a C^*-algebra A: isomorphisms of one ideal in A onto another [5]. He has shown that many interesting C^*-algebras can be viewed as crossed products by partial automorphisms, and that these crossed products have much in common with ordinary crossed products by actions of $\mathbb{Z}$. McClanahan subsequently extended Exel's ideas to cover *partial actions* of more general groups by partial automorphisms, and showed that, rather surprisingly, many important results on crossed products by free groups carry over to crossed products by partial actions [9].

Here we give a characterisation of (reduced) crossed products by partial actions of discrete groups, which is similar in spirit to that of Landstad for ordinary crossed products (see [7] or [12, 7.8.8]), and which both generalises and simplifies Exel's characterisation of crossed products by single partial automorphisms [5, Theorem 4.21]. Our main result says that a C^*-algebra B is a crossed product by a partial action of G if and only if it carries a coaction δ of G and there is a *partial representation* of G by partial isometries in the double dual B^{**} which induces suitable isomorphisms among the spectral subspaces of δ; this result takes a particularly elegant form when G is the free group $\mathbb{F}_n$. We then use our classification to identify the Cuntz algebras $\mathcal{O}_n$, the

Date: February 13, 1996.

1991 *Mathematics Subject Classification.* Primary 46L55.

This research was partially supported by the Australian Research Council and by the National Science Foundation (under Grant No. DMS9401253).

Cuntz-Krieger algebras $\mathcal{O}_A$, and the Toeplitz or Wiener-Hopf algebras of Nica [10] as crossed products by partial actions of $\mathbb{F}_n$. We have not previously seen the canonical coaction of $\mathbb{F}_n$ on $\mathcal{O}_n$ used in a serious way, so these ideas may have interesting implications for the study of coactions of discrete groups.

We begin with a discussion of partial actions and covariant representations. A partial action α of G on A is a collection $\{D_s : s \in G\}$ of ideals in A, and isomorphisms α_s of $D_{s^{-1}}$ onto D_s, such that α_{st} extends $\alpha_s \circ \alpha_t$ from its natural domain $D_{t^{-1}} \cap \alpha_t^{-1}(D_{s^{-1}})$. Calculations involving the domains can be tricky, so we have taken care to make the various relationships explicit. A partial representation of G is a map u of G into the set of partial isometries on a Hilbert space (or in a C^*-algebra) such that u_{st} extends $u_s u_t$, and a covariant representation (π, u) of (A, G, α) consists of a representation π of A and a partial representation u of G such that $\pi(\alpha_s(a)) = u_s \pi(a) u_s^*$ for $a \in D_{s^{-1}}$. McClanahan did not discuss partial representations in their own right, so we have included a detailed discussion of them and their relationship to covariant representations.

A key technical innovation in our treatment is the implementation of Hilbert-module isomorphisms of spectral subspaces by multipliers of imprimitivity bimodules, as introduced in [4]; the particular multipliers involved here will form the partial representation of G in the double dual of the crossed product. We therefore recall in §2 some facts about multipliers of bimodules, relate them to Hilbert-module isomorphisms, and discuss how in certain situations the whole structure can be embedded in the double dual of a C^*-algebra.

In §3, we construct the crossed product $A \times_\alpha G$ of a partial action α, as the C^*-algebra generated by a universal covariant representation of (A, G, α) in $(A \times_\alpha G)^{**}$. Associated to any faithful representation π of A is a regular representation of $A \times_\alpha G$; up to isomorphism, the image is independent of the choice of π, and is called the reduced crossed product $A \times_{\alpha,r} G$. Of course, both crossed products turn out to be the ones studied in [9], but our emphasis on universal properties allows us to see quickly that they carry a dual coaction of G. Our characterisation of the reduced crossed product in terms of this dual coaction is Theorem 4.1.

An ordinary action α of the free group $\mathbb{F}_n$ is determined completely by the n automorphisms α_{g_i} corresponding to generators $\{g_i\}$ of $\mathbb{F}_n$. It is quite easy to construct partial actions of $\mathbb{F}_n$ from n partial automorphisms [9, Example 2.3], but in general even partial actions of $\mathbb{F}_1 = \mathbb{Z}$ need not arise this way. So we concentrate in §5 on a family of partial

actions α of $\mathbb{F}_n$ which are determined by $\{\alpha_{g_i}\}$; crossed products by such *multiplicative* partial actions can be characterised in terms of the spectral subspaces of the dual coaction corresponding to the generators of $\mathbb{F}_n$. The main result here is Theorem 5.6, and its applications to Cuntz algebras, Cuntz-Krieger algebras and Nica's Toeplitz algebras are the content of our last section.

Acknowledgements. As this paper was being written up, the authors received a copy of [6], in which Exel proves a result related to our Theorem 4.1. However, his result concerns C^*-algebraic bundles, and uses techniques substantially different from ours.

The bulk of this research was carried out while the first author was visiting the University of Newcastle. He wishes to thank Iain Raeburn for his hospitality. The authors thank Marcelo Laca for helpful conversations, and for introducing them to the work of Nica through a collaboration with the second author [8].

1. Partial actions and covariant representations

Definition 1.1. A *partial action* of a discrete group G on a C^*-algebra A consists of a collection $\{D_s\}_{s\in G}$ of closed ideals of A and isomorphisms $\alpha_s\colon D_{s^{-1}} \to D_s$ such that

(i) $D_e = A$;
(ii) α_{st} extends $\alpha_s\alpha_t$ for all $s,t \in G$ (where the domain of $\alpha_s\alpha_t$ is $\alpha_t^{-1}(D_{s^{-1}})$, which is by definition contained in $D_{t^{-1}}$).

We suppose for the rest of this section that α is a partial action of G on A. We shall frequently need to intersect domains of the partial isomorphims, and shall use without comment that the intersection of two ideals I, J in a C^*-algebra is the ideal $IJ := \overline{\mathrm{sp}}\{ij : i \in I, j \in J\}$. We shall also need to recall that the double dual I^{**} of an ideal I naturally embeds as an ideal in A^{**}.

Lemma 1.2. *For $s,t \in G$ we have:*

(i) $\alpha_e = \iota$, *and* $\alpha_{s^{-1}} = \alpha_s^{-1}$;
(ii) $\alpha_s(D_{s^{-1}}D_t) = D_sD_{st}$;
(iii) $\alpha_s \circ \alpha_t$ *is an isomorphism of* $D_{t^{-1}}D_{t^{-1}s^{-1}}$ *onto* D_sD_{st}.

Remark 1.3. Suppose instead of (ii) we had

$$\alpha_s(D_{s^{-1}}D_t) \subset D_{st} \quad \text{and} \quad \alpha_{st}(x) = \alpha_s\alpha_t(x) \text{ for } x \in D_{t^{-1}}D_{t^{-1}s^{-1}},$$

so that α is a partial action in the sense of McClanahan. Then α would satisfy (ii), and hence be a partial action in our sense. To see this, just

note that

$$\operatorname{dom}(\alpha_s \circ \alpha_t) = \alpha_t^{-1}(D_{s^{-1}}) = \alpha_t^{-1}(D_t D_{s^{-1}}) \subset D_{t^{-1}} D_{t^{-1}s^{-1}},$$

so the equation $\alpha_{st} = \alpha_s \circ \alpha_t$ on $D_{t^{-1}} D_{t^{-1}s^{-1}}$ says what we need. Indeed, by part (ii) of the lemma, this says *exactly* what we need, so our definition is equivalent to McClanahan's.

Proof of Lemma 1.2. For part (i), observe that $\alpha_e\alpha_e$ is defined everywhere and equal to α_e, which is therefore the identity transformation on all of A. Because α_e extends $\alpha_s\alpha_{s^{-1}}$ and $\alpha_{s^{-1}}\alpha_s$, this forces $\alpha_{s^{-1}} = \alpha_s^{-1} : D_s = D_{(s^{-1})^{-1}} \to D_{s^{-1}}$. For (ii), we note that $\alpha_{t^{-1}}(D_{s^{-1}})$ is by definition $\alpha_{t^{-1}}(D_t D_{s^{-1}})$, and because α_{st} extends $\alpha_s\alpha_t$, we have

$$\alpha_{t^{-1}}(D_t D_{s^{-1}}) \subset D_{t^{-1}} D_{t^{-1}s^{-1}} \quad \text{for all } s, t. \tag{1.1}$$

Applying this with t^{-1} replaced by t shows

$$\alpha_t\big(D_{t^{-1}} D_{t^{-1}s^{-1}}\big) \subset D_t D_{t(t^{-1}s^{-1})} = D_t D_{s^{-1}},$$

and because $\alpha_{t^{-1}} = \alpha_t^{-1}$, this implies that we must have equality in (1.1). Since the left-hand side of (1.1) is the natural domain of $\alpha_s\alpha_t$, and the range of $\alpha_s\alpha_t$ is the natural domain of $\alpha_t^{-1}\alpha_s^{-1} = \alpha_{t^{-1}}\alpha_{s^{-1}}$, part (iii) follows. □

Definition 1.4. For $s \in G$, we let p_s denote the projection in A^{**} which is the identity of D_s^{**}.

The projections p_s belong to the center of A^{**}, and p_s is the weak* limit of any bounded approximate identity for D_s. We always have $p_s \in M(D_s)$, but p_s may not be in $M(A)$, as shown by the following example.

Example 1.5. Let $A = C_0(0, \infty)$ and $G = \mathbb{Z}_2$. Define $D_1 = \{f \in A \mid f(x) = 0 \text{ for } x \leq 1\}$, and let α_1 be the identity map of D_1. Then p_1 is the characteristic function of $(1, \infty)$, which is not in $M(A)$ because it is not continuous on $(0, \infty)$.

To define covariant representations of partial actions, we need an appropriate notion of *partial representations* of groups by partial isometries. The idea is that u_{st} should extend u_su_t; for this to make sense, u_su_t must be a partial isometry, so we insist that the range projections commute. The following Lemma describes what we mean by "v extends u", and is presumably standard.

Lemma 1.6. *We define a relation $\preceq$ on the set of partial isometries on a Hilbert space $\mathcal{H}$ by*

$$u \preceq v \iff uu^* = uv^*.$$

Then $u \preceq v$ precisely when the initial space $u^*u(\mathcal{H})$ of u is contained in $v^*v(\mathcal{H})$, and $v = u$ on $u^*u(\mathcal{H})$; we have $u \preceq v \iff u^*u = v^*u$, and $\preceq$ is a partial order on the set of partial isometries on $\mathcal{H}$.

Proof. Since uu^* is self-adjoint, $uu^* = uv^*$ implies $uu^* = vu^*$. But then $vu^*u = (uu^*)u = u$ implies that $v = u$ on the range of u^*u. In particular, this implies that v^* maps the range of u into the initial space of u, and hence $uu^* = uv^*$ implies

$$u^*u = u^*(uu^*)u = u^*(uv^*)u = (u^*u)v^*u = v^*u.$$

Conversely, $u^*u = v^*u$ implies that $v^* = u^*$ on the range of uu^*, and hence that v maps the initial space of u into the range of u; thus

$$uu^* = u(u^*u)u^* = u(u^*v)u^* = (uu^*)vu^* = vu^*.$$

Finally, since $u \preceq v$ implies $uv^*v = u$, it is easy to check that $\preceq$ is transitive, and that $u \preceq v$, $v \preceq u$ force $u = v$. □

Definition 1.7. A *partial representation* of G on a Hilbert space $\mathcal{H}$ is a map $u\colon G \to \mathcal{B}(\mathcal{H})$ such that the u_s are partial isometries with commuting range projections, and

$$u_e u_e^* = 1; \tag{1.2}$$

$$u_s^* u_s = u_{s^{-1}} u_{s^{-1}}^* \qquad \text{for all } s \in G; \tag{1.3}$$

$$u_s u_t \preceq u_{st} \qquad \text{for all } s, t \in G. \tag{1.4}$$

We begin by listing some straightforward consequences of the definition.

Lemma 1.8. *If u is a partial representation, then*

$$u_e = 1; \tag{1.5}$$

$$u_s^* = u_{s^{-1}} \qquad \textit{for all } s \in G; \tag{1.6}$$

$$u_s u_t = u_s u_s^* u_{st} \qquad \textit{for all } s, t \in G. \tag{1.7}$$

Proof. For (1.5), note that (1.2) and (1.4) imply that u_e is an idempotent coisometry. Since u_s^* and $u_{s^{-1}}$ are partial isometries with the same range projection, and $u_s u_{s^{-1}} \preceq 1$, we have (1.6). For the last part, we use the relation $u \preceq v$ in the forms $u = uu^*v$, $uu^* = uv^*$, and then $v^*u = u^*u$, to deduce that

$$\begin{aligned} u_s u_t &= (u_s u_t)(u_s u_t)^* u_{st} = (u_s u_t u_t^* u_s^*)(u_s u_s^* u_{st}) \\ &= (u_s u_t u_{st}^*)(u_s u_s^* u_{st}) = u_s(u_t u_{st}^* u_s) u_s^* u_{st} \\ &= u_s(u_s^* u_{st} u_{st}^* u_s) u_s^* u_{st}, \end{aligned}$$

which equals $u_s u_s^* u_{st}$ because $u_s^* u_{st}$ is a partial isometry. □

Remark 1.9. Conditions (1.5)–(1.7) are stronger forms of (1.2)–(1.4); to see this for (1.4), note that $u_s u_s^*$ is a projection commuting with $u_{st}u_{st}^*$, and hence

$$(u_s u_t)u_{st}^* = u_s u_s^* u_{st} u_{st}^* = (u_s u_s^* u_{st})(u_{st}^* u_s u_s^*) = (u_s u_t)(u_s u_t)^*.$$

Definition 1.10. Let α be a partial action of G on A. A *covariant representation* of (A, G, α) is a pair (π, u) consisting of a nondegenerate representation π of A and a partial representation u of G on the same Hilbert space, satisfying

$$u_s u_s^* = \pi(p_s); \tag{1.8}$$

$$\pi(\alpha_s(a)) = u_s \pi(a) u_s^* \quad \text{for } a \in D_{s^{-1}}. \tag{1.9}$$

Lemma 1.11. *Let α be a partial action of G on A, let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a nondegenerate representation, and let $u\colon G \to \mathcal{B}(\mathcal{H})$ be a map satisfying* (1.8)–(1.9)*. Then the following are equivalent*:

(i) *u is a partial representation* (*and* (π, u) *is a covariant representation*);
(ii) *$u_s u_t \preceq u_{st}$ for all $s, t \in G$;*
(iii) *$\pi(p_{st})u_s u_t = \pi(p_s)u_{st}$ for all $s, t \in G$;*
(iv) *$\pi(a)u_s u_t = \pi(a)u_{st}$ for all $s, t \in G, a \in D_s D_{st}$.*

Proof. Note first that by (1.8)–(1.9), the u_s are partial isometries with commuting range projections, and $u_e u_e^* = \pi(p_e) = 1$, so (1.2) holds. Further, (iii) is equivalent to (iv) since $p_s p_{st}$ is the identity of $(D_s D_{st})^{**}$ in A^{**}, and Lemma 1.8 tells us that (i) implies (iii).

Assume (ii). We will show (1.6), giving (1.3), hence (i). First note that (1.5) holds, since its proof only required (1.2) and (1.4). We have

$$\begin{aligned} u_s u_s^* &= \pi(p_s) = \pi \circ \alpha_s(p_{s^{-1}}) = \operatorname{Ad} u_s \circ \pi(p_{s^{-1}}) \\ &= u_s u_{s^{-1}} u_{s^{-1}}^* u_s^* = u_s u_{s^{-1}} u_{ss^{-1}}^* = u_s u_{s^{-1}}, \end{aligned}$$

so that

$$u_s \preceq u_{s^{-1}}^*. \tag{1.10}$$

Since the partial ordering $\preceq$ on partial isometries is conjugation-invariant, we get $u_s^* \preceq u_{s^{-1}}$. Applying (1.10) with s replaced by s^{-1}, we arrive at

$$u_s^* \preceq u_{s^{-1}} \preceq u_s^*,$$

so $u_s^* = u_{s^{-1}}$, which is (1.6).

Finally, assume (iii). To show (ii), we again need (1.5) and (1.6). (1.5) follows from (1.2) and (iii). For (1.6), we have

$$u_s u_{s^{-1}} = \pi(p_{ss^{-1}})u_s u_{s^{-1}} = \pi(p_s)u_{ss^{-1}} = u_s u_s^*,$$

and the argument of the preceding paragraph shows $u_{s^{-1}} = u_s^*$. We now show (ii):

$$\begin{aligned} u_s u_t u_t^* u_s^* &= u_s \pi(p_{s^{-1}})\pi(p_t)u_s^* = \pi \circ \alpha_s(p_{s^{-1}}p_t) = \pi(p_s p_{st}) \\ &= \pi(p_s p_{st})u_{st}u_{st}^* = \pi(p_s p_{st})u_s u_t u_{st}^* \\ &= u_s \pi(p_{s^{-1}}p_t)u_s^* u_s u_t u_{st}^* = u_s u_t u_{st}^*. \quad \square \end{aligned}$$

Remark 1.12. The previous Lemma shows that our definition of covariant representation is equivalent to McClanahan's [9]. So ours is actually a slight improvement over McClanahan's, in that the conditions $u_s^* u_s = \pi(p_{s^{-1}})$ and $u_s^* = u_{s^{-1}}$ follow automatically.

2. Multipliers of imprimitivity bimodules

Recall from [4] that if X is a $C - D$ imprimitivity bimodule, a *multiplier* of X is a pair $m = (m_C, m_D)$, where $m_C \in \mathcal{L}_C(C, X)$ and $m_D \in \mathcal{L}_D(D, X)$ satisfy

$$m_C(c) \cdot d = c \cdot m_D(d) \quad \text{for} \quad c \in C, d \in D.$$

(Actually, [4, Lemma 1.4] shows that adjointability of m_C and m_D is automatic.) The set $M(X)$ of multipliers of X is called the *multiplier bimodule*; with

$$c \cdot m = m_C(c) \quad \text{and} \quad m \cdot d = m_D(d),$$

$M(X)$ becomes a $C - D$ bimodule containing X. The module actions of C and D on $M(X)$ extend to $M(C)$ and $M(D)$, and the C- and D-valued inner products on X extend to $M(C)$- and $M(D)$-valued inner products on $M(X)$, which we continue to denote by ${}_C\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_D$. For $x \in X$ and $m \in M(X)$ we have

$$(2.1) \qquad {}_C\langle x, m \rangle = m_C^*(x) \quad \text{and} \quad \langle m, x \rangle_D = m_D^*(x).$$

Lemma 2.1. *Let X be a $C - D$ imprimitivity bimodule.*

(i) *There is a left Hilbert C-module isomorphism ψ of X onto C if and only if there is a multiplier $m = (m_C, m_D)$ of X such that*

$$(2.2) \qquad {}_C\langle m, m \rangle = 1_{M(C)}, \quad \langle m, m \rangle_D = 1_{M(D)}$$

and $\psi(x) = {}_C\langle x, m \rangle = m_C^*(x)$ for all $x \in X$.

(ii) *Let m be a multiplier of X satisfying (2.2), and let $\psi = m_C^* : X \to C$ and $\phi = m_D^* : X \to D$ be the corresponding isomorphisms of Hilbert modules. Then there is a C^*-algebra isomorphism α of D onto C such that $\alpha(d) = {}_C\langle m \cdot d, m \rangle$ and $\psi = \alpha \circ \phi$. We write $\alpha = \operatorname{Ad} m$.*

Proof. Since ψ preserves the C-valued inner products, the inverse $\psi^{-1} : C \to X$ is an adjoint for ψ, and ψ^{-1} itself is in $\mathcal{L}_C(C, X)$. Thus by [4, Proposition 1.3] there is a multiplier $m = (m_C, m_D)$ with $m_C = \psi^{-1}$, and then ${}_C\langle x, m\rangle = \psi(x)$ for all $x \in X$. Thus for $x \in X$ we have

$$\begin{aligned} x &= \psi^{-1} \circ \psi(x) = \psi^{-1}\big({}_C\langle x, m\rangle\big) = m_C\big({}_C\langle x, m\rangle\big) \\ &= {}_C\langle x, m\rangle \cdot m = x \cdot \langle m, m\rangle_D, \end{aligned}$$

and for $c \in C$ we have

$$c = \psi \circ \psi^{-1}(c) = {}_C\langle \psi^{-1}(c), m\rangle = {}_C\langle c \cdot m, m\rangle = c_C\langle m, m\rangle,$$

from which (2.2) follows. It is easy to check that, given $m \in M(X)$ satisfying (2.2), $\psi : x \mapsto {}_C\langle x, m\rangle$ is a Hilbert module isomorphism with inverse $c \mapsto c \cdot m$.

For part (ii), define $\alpha = \psi \circ \phi^{-1}$, and note that $\alpha(d) = {}_C\langle m \cdot d, m\rangle$. Then α is a linear isomorphism of D onto C; to see it is in fact a C^*-isomorphism requires only calculations using the properties of m. For example, for $a, b \in D$ we have

$$\begin{aligned} {}_C\langle m \cdot ab, m\rangle &= {}_C\langle m \cdot a\langle m, m\rangle_D\, b, m\rangle = {}_C\langle {}_C\langle m \cdot a, m\rangle m \cdot b, m\rangle \\ &= {}_C\langle m \cdot a, m\rangle\, {}_C\langle m \cdot b, m\rangle. \qquad \square \end{aligned}$$

In the above proof we could have constructed α directly from ψ: a Hilbert module isomorphism induces an isomorphism of the algebras of compact operators, and hence $D = \mathcal{K}_C(X) \cong \mathcal{K}_C(C) = C$.

Proposition 2.1 simplifies part of [5, Section 4], where both ψ and α are postulated, and ψ is postulated to be a $C{-}D$ bimodule isomorphism rather than just a left C-module isomorphism. Our approach also justifies Exel's feeling that one of his results [5, Proposition 4.13] is a kind of "partial multiplier" property: he is really using a module multiplier in the sense of [4].

Now suppose B is a C^*-algebra and X is a closed subspace of B such that $XX^*X \subset X$. Let $C = XX^*$ (caution: we use Exel's convention that this denotes the *closed* linear span of the set of products!) and $D = X^*X$. Then C and D are C^*-subalgebras of B, and X is a $C - D$ imprimitivity bimodule. Let p and q be the identities of C^{**} and D^{**}, respectively, regarded as projections in B^{**}. [4, Proposition 2.4] implies that we can identify $M(X)$ with

$$\{m \in pB^{**}q \mid Cm \cup mD \subset X\},$$

in such a way that the module actions are given by multiplication in B^{**}, and the inner products are given by, for example, ${}_C\langle m, n\rangle = mn^*$. Fortunately, there is no ambiguity between the various meanings of m^*:

the adjoint m^* of m in B^{**} implements the adjoint of the module homomorphism $c \mapsto c \cdot m$, which is given by $x \mapsto {}_C\langle x, m\rangle = xm^*$. Finally we observe that, if ρ is any faithful nondegenerate representation of B on $\mathcal{H}$, then the canonical extension of ρ to a normal representation of B^{**} on $\mathcal{H}$ maps $M(X)$ isomorphically onto

$$\{x \in \rho(p)\mathcal{B}(\mathcal{H})\rho(q) \mid \rho(C)x \cup x\rho(D) \subset \rho(X)\}.$$

3. Crossed products and the dual coaction

Suppose α is a partial action of a group G on a C^*-algebra A, and (π, u) is a covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$. Let

$$C^*(\pi, u) = \overline{\sum_{s\in G} \pi(D_s)u_s}.$$

A short computation using the relation $\alpha_s(D_{s^{-1}}D_t) \subset D_{st}$ shows that for $s, t \in G$, $a \in D_s$ and $b \in D_t$ we have

$$\pi(a)u_s\pi(b)u_t = \pi \circ \alpha_s(\alpha_{s^{-1}}(a)b)u_{st}$$

and

$$(\pi(a)u_s)^* = \pi \circ \alpha_{s^{-1}}(a^*)u_s^*,$$

so the closed subspace $C^*(\pi, u)$ of $B(\mathcal{H})$ is a C^*-algebra, which we call the *C^*-algebra of the covariant representation* (π, u).

We would like to define the crossed product $A \times_\alpha G$ to be the C^*-algebra of a universal covariant representation (π, u). The image $\pi(A)$ is a C^*-subalgebra of $C^*(\pi, u)$, but in general (as can be seen from Example 1.5 and Proposition 3.5) the partial isometries u_s need not be multipliers of $C^*(\pi, u)$. So to get a suitable universal covariant representation, we shall have to work in the double dual $(A \times_\alpha G)^{**}$, and we shall have to construct the algebra $A \times_\alpha G$ as an enveloping algebra.

Following McClanahan's development, let L_c denote the vector space of functions $f : G \to A$ of finite support such that $f(s) \in D_s$ for all $s \in G$. For $a \in D_s$, let

$$F(a, s)(t) = \begin{cases} a & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases}$$

Then L_c is the linear span of the $F(a, s)$. (McClanahan writes $a\delta_s$ for our $F(a, s)$; we avoid the notation δ_s because it has other connotations for coaction freaks, and call them m_s instead. The precise meaning

of m_s will be made clear shortly.) The *-algebra structure of L_c is determined by

$$F(a,s)F(b,t) = F(\alpha_s(\alpha_s^{-1}(a)b), st), \text{ and}$$
$$F(a,s)^* = F(\alpha_s^{-1}(a^*), s^{-1}).$$

Note that A embeds as a *-subalgebra of L_c via $a \mapsto F(a,e)$, so any *-representation Π of L_c on a Hilbert space restricts to a representation of the C^*-algebra A. Hence

$$\begin{aligned}\|\Pi(F(a,s))\|^2 &= \|\Pi(F(a,s))^*\Pi(F(a,s))\|\\ &= \|\Pi(F(\alpha_s^{-1}(a^*), s^{-1})F(a,s))\|\\ &= \|\Pi(F(\alpha_s^{-1}(a^*a), e))\| \le \|\alpha_s^{-1}(a^*a)\| = \|a\|^2.\end{aligned} \tag{3.1}$$

We deduce that the greatest C^*-seminorm on L_c is finite (it is in fact a norm, although we do not need this), so it makes sense to define the crossed product $A \times_\alpha G$ as the C^*-completion of L_c. The double dual A^{**} embeds naturally in $(A \times_\alpha G)^{**}$, so the projections p_s are naturally identified with (no longer central) projections in $(A \times_\alpha G)^{**}$.

Let

$$X_s = \{F(a,s) \mid a \in D_s\}.$$

The calculation (3.1) shows that this is a closed subspace of $A \times_\alpha G$, and $X_s X_s^* = D_s$ and $X_s^* X_s = D_{s^{-1}}$ as subalgebras of $A \subset A \times_\alpha G$, so X_s is a $D_s - D_{s^{-1}}$ imprimitivity bimodule. The discussion at the end of §2 shows that

$$M(X_s) = \{m \in p_s(A \times_\alpha G)^{**}p_{s^{-1}} \mid D_s m \cup m D_{s^{-1}} \subset X_s\}. \tag{3.2}$$

A calculation shows that the maps $l : D_s \to X_s$, $r : D_{s^{-1}} \to X_s$ defined by

$$l(a) = F(a,s), \quad r(b) = F(\alpha_s(b), s)$$

satisfy $l(a) \cdot b = a \cdot r(b)$, and hence define a multiplier $m_s := (l,r)$ of $M(X_s)$ [4, Lemma 1.4]. The identification (3.2) allows us to view m_s as an element of $p_s(A \times_\alpha G)^{**}p_{s^{-1}}$, which by definition satisfies

$$a \cdot m_s = F(a,s) \text{ for } a \in D_s, \text{ and } m_s \cdot b = F(\alpha_s(b), s) \text{ for } b \in D_{s^{-1}}.$$

Recall that the $M(D_s)$-valued inner product is given on $p_s(A \times_\alpha G)^{**}p_{s^{-1}}$ by ${}_{D_s}\langle m, n\rangle = mn^*$; thus for $a \in D_s$ we have

$$a(m_s m_s^*) = a\,{}_{D_s}\langle m_s, m_s\rangle = {}_{D_s}\langle a \cdot m_s, m_s\rangle = {}_{D_s}\langle F(a,s), m_s\rangle.$$

It follows from (2.1) that ${}_{D_s}\langle F(a,s), m_s\rangle$ is given in terms of the adjoint of $l : D_s \to X_s$ by $l^*(F(a,s))$, which by an easy calculation is seen to be a. Thus $a(m_s m_s^*) = a$ for all $a \in D_s$, and $m_s m_s^* = 1_{M(D_s)} = p_s$. Similiarly, we can verify that $m_s^* m_s = p_{s^{-1}}$. The multipliers m_s therfore

have the property (2.2) of Lemma 2.1, and induce isomorphisms $\operatorname{Ad} m_s$ of $D_{s^{-1}}$ onto D_s, characterised by $\operatorname{Ad} m_s(a) = {}_{D_s}\langle m_s \cdot a, m_s\rangle$. Another calculation shows that

$$\operatorname{Ad} m_s(a) = {}_{D_s}\langle m_s \cdot a, m_s\rangle = l^*(m_s \cdot a) = l^*\big(F(\alpha_s(a), s)\big) = \alpha_s(a),$$

so the elements m_s of $(A \times_\alpha G)^{**}$ are partial isometries implementing the partial automorphisms α_s.

We claim that the inclusion $\iota\colon A \hookrightarrow A \times_\alpha G$ and m form a covariant representation (ι, m) of (A, G, α) in $(A \times_\alpha G)^{**}$. To see this, let $a \in D_s D_{st}$, and let e_i be an approximate identity for D_t. Then in the weak* topology of $(A \times_\alpha G)^{**}$, we have

$$\begin{aligned} a m_s m_t &= \lim a m_s e_i m_t = \lim \alpha_s(\alpha_{s^{-1}}(a) e_i) m_{st} \\ &= \alpha_s(\alpha_{s^{-1}}(a)) m_{st} = a m_{st}. \end{aligned}$$

It now follows from Lemma 1.11 that (ι, m) is covariant, as claimed.

Next, let (π, u) be a covariant representation of (A, G, α). Then (π, u) defines a representation of L_c, which extends to a representation $\pi \times u$ of $A \times_\alpha G$ by definition of the enveloping norm, and hence also to a normal representation, also denoted $\pi \times u$, of $(A \times_\alpha G)^{**}$. Applying $\pi \times u$ to $\iota(a) = F(a, e)$ gives $\pi(a)$, and for $a \in D_{s^{-1}}$ we have

$$\begin{aligned} \pi \times u(m_s)(\pi(a)h) &= \pi \times u\big((m_s a)h\big) = \pi \times u\big(F(\alpha_s(a), s)\big)h \\ &= \pi(\alpha_s(a)) u_s h = u_s \pi(a) u_s^* u_s h = u_s \pi(a) \pi(p_s) h \\ &= u_s \pi(a p_s) h = u_s(\pi(a)h), \end{aligned}$$

so $\pi \times u(m_s) = u_s$. Thus

$$(\pi \times u) \circ \iota = \pi \quad \text{and} \quad (\pi \times u) \circ m = u. \tag{3.3}$$

If we want to avoid extending to $(A \times_\alpha G)^{**}$, we need to rewrite (3.3) as

$$(\pi \times u)(a m_s) = \pi(a) u_s \quad \text{for} \quad a \in D_s.$$

That covariant representations extend to $A \times_\alpha G$ in this way characterises the crossed product:

Proposition 3.1. *Let (π, u) be a covariant representation of (A, G, α). Then $\pi \times u$ is faithful if and only if for every covariant representation (ρ, v) there exists a homomorphism $\theta\colon C^*(\pi, u) \to C^*(\rho, v)$ such that*

$$\theta(\pi(a) u_s) = \rho(a) v_s \quad \textit{for} \quad a \in D_s.$$

Proof. Applying the hypothesis to the canonical covariant representation (ι, m) of (A, G, α) in $(A \times_\alpha G)^{**}$ gives an inverse θ for $\pi \times u$. $\square$

Although this is not a deep result, it does give the easiest way of recognising the full crossed product. However, it is difficult to see how to turn this into a convenient categorical definition of the crossed product, as is often done for ordinary actions. For example, even if $\pi \times u$ is faithful on $A \times_\alpha G$, we do not know if the canonical extension to a normal representation of $(A \times_\alpha G)^{**}$ is faithful on the C^*-algebra generated by A and m_G. Thus for all we know, $C^*(A \cup m_G)$ and $C^*(\pi(A) \cup u_G)$ could be essentially different even though $A \times_\alpha G$ and $C^*(\pi, u)$ are isomorphic. For a partial solution of this conundrum, see Proposition 3.3 below.

Proposition 3.2. *Let α be a partial action of a discrete group G on a C^*-algebra A. Then there is a unique coaction $\hat{\alpha}$ of G on $A \times_\alpha G$ such that $\hat{\alpha}(am_s) = am_s \otimes s$ for $a \in D_s$.*

Proof. Define maps π and u from A and G, respectively, to $(A \times_\alpha G)^{**} \otimes C^*(G)$ by

$$\pi(a) = a \otimes 1 \quad \text{and} \quad u_s = m_s \otimes s.$$

Then u is a partial representation (being a tensor product of two such). We have

$$u_s u_s^* = m_s m_s^* \otimes 1 = p_s \otimes 1 = \pi(p_s),$$
$$u_s u_t u_t^* u_s^* = m_s m_t m_t^* m_s^* \otimes stt^{-1}s^{-1} = m_s m_t m_{st}^* \otimes 1 = u_s u_t u_{st}^*,$$

and

$$\begin{aligned}\operatorname{Ad} u_s \circ \pi(a) &= \operatorname{Ad} m_s(a) \otimes 1 \\ &= \alpha_s(a) \otimes 1 = \pi \circ \alpha_s(a) \qquad \text{for } a \in D_{s^{-1}},\end{aligned}$$

so (π, u) is a covariant representation of (A, G, α) by Lemma 1.11(ii). Clearly $C^*(\pi, u) \subset (A \times_\alpha G) \otimes C^*(G)$.

Define $\theta\colon (A \times_\alpha G) \otimes C^*(G) \to A \times_\alpha G$ by $\theta(x \otimes s) = x$ for $s \in G$. This is well-defined on the minimal tensor product because it is the tensor product of the identity homomorphism of $A \times_\alpha G$ and the augmentation representation of $C^*(G)$. We have

$$\theta \circ (\pi \times u)(am_s) = \theta(am_s \otimes s) = am_s \quad \text{for} \quad a \in D_s.$$

Hence, $\hat{\alpha} = \pi \times u$ is a faithful homomorphism of $A \times_\alpha G$. Nondegeneracy of $\hat{\alpha}$ as a homomorphism into $(A \times_\alpha G) \otimes C^*(G)$ and the coaction identity are obvious. □

As a first application, we use the dual coaction $\hat{\alpha}$ to show that in a faithful representation $\pi \times u$, u is as faithful as m is:

Proposition 3.3. *If $\pi \times u$ is a faithful representation of $A \times_\alpha G$, then $u_s = u_t$ implies $m_s = m_t$.*

Proof. Suppose $u_s = u_t$, and suppose first that $D_sD_t \neq \{0\}$. Then for any nonzero $a \in D_sD_t$ we have

$$(\pi \times u)(am_s) = \pi(a)u_s = \pi(a)u_t = (\pi \times u)(am_t),$$

so $am_s = am_t$ by hypothesis. Since $a \neq 0$ implies $am_s \neq 0$, applying the dual coaction $\hat{\alpha}$ gives $am_s \otimes s = am_t \otimes t$, so s and t are linearly dependent elements of $C^*(G)$, forcing $s = t$ and $m_s = m_t$. If $D_s \cap D_t = D_sD_t = \{0\}$, then $p_sp_t = 0$, and

$$u_su_s^* = u_su_s^*u_tu_t^* = \pi(p_s)\pi(p_t) = \pi(p_sp_t) = 0,$$

forcing $u_s = 0$. But then $\pi \times u(am_s) = \pi(a)u_s = 0$ for all $a \in D_s$, and because $\pi \times u$ is faithful this implies $D_s = \{0\}$ and $m_s = 0$. Simlarly, $u_t = u_s = 0$ implies $m_t = 0$, so again we have $m_s = m_t$, as required. □

Recall from [15] that if δ is a coaction of G on a C^*-algebra B, then for $s \in G$ the associated spectral subspace is

$$B_s = \{b \in B \mid \delta(b) = b \otimes s\}.$$

If χ_s is the characteristic function of $\{s\}$, regarded as an element of $B(G) = C^*(G)^*$, then $\delta_s = (\iota \otimes \chi_s) \circ \delta$ is a projection of B onto B_s.

Proposition 3.4. *The spectral subspaces for the dual coaction $\hat{\alpha}$ on a partial crossed product $A \times_\alpha G$ are given by $(A \times_\alpha G)_s = D_sm_s = m_sD_{s^{-1}}$.*

Proof. Let $\hat{\alpha}_s\colon A \times_\alpha G \to (A \times_\alpha G)_s$ be the canonical projection. Clearly $D_sm_s \subset (A \times_\alpha G)_s$. On the other hand, any $x \in (A \times_\alpha G)_s$ can be approximated by a finitely nonzero sum $\sum_t a_tm_t$, and then

$$x = \hat{\alpha}_s(x) \approx \hat{\alpha}_s(\sum_t a_tm_t) = a_sm_s.$$

Hence $(A \times_\alpha G)_s \subset D_sm_s$, proving the first equality. The second equality follows from the covariance of (ι, m):

$$m_sD_{s^{-1}} = m_sD_{s^{-1}}p_{s^{-1}} = m_sD_{s^{-1}}m_s^*m_s = \alpha_s(D_{s^{-1}})m_s = D_sm_s. \quad \square$$

Proposition 3.5. *For $s \in G$, consider the following conditions*:

(i) $p_s \in M(A)$;
(ii) $D_s = Ap_s$;
(iii) $M(D_s) \subset M(A)$;
(iv) $m_s \in M(A \times_\alpha G)$.

Conditions (i)–(iii) *are equivalent, and are implied by* (iv). *Moreover, if* (i) *holds for both s and s^{-1}, then* (iv) *holds as well.*

Proof. That (i) implies (ii) must be a well-known general fact about ideals of C^*-algebras, but we lack a reference. Let A act via its universal representation on $\mathcal{H}$. It suffices to show that any state ω of A annihilating D_s also annihilates Ap_s. There exists $\xi \in \mathcal{H}$ such that $\omega(a) = (a\xi, \xi)$. Since p_s is in the weak* closure of D_s, $(p_s\xi, \xi) = 0$. This forces $p_s\xi = 0$, so for any $a \in A$ we have $\omega(ap_s) = (ap_s\xi, \xi) = 0$, as required.

The chain (ii) implies (iii) implies (i) is routine.

Assuming (iv), we have $p_s = m_s m_s^* \in M(A \times_\alpha G)$ also. Since $\hat{\alpha}(p_s) = p_s \otimes 1$, we get

$$p_s \in M(A \times_\alpha G)_e = M((A \times_\alpha G)_e) = M(D_e m_e) = M(A).$$

Finally, assume (i) holds for both s and s^{-1}. Since $A \times_\alpha G = \overline{\sum_t D_t m_t}$ and $m_s^* = m_{s^{-1}}$, (iv) follows from the following computation for $a \in D_t$:

$$\begin{aligned} a m_t m_s &= m_t \alpha_{t^{-1}}(a) p_s m_s \\ &= \alpha_t(\alpha_{t^{-1}}(a) p_s) m_t m_s && \text{since } \alpha_{t^{-1}}(a) p_s \in D_{t^{-1}} D_s \\ &= \alpha_t(\alpha_{t^{-1}}(a) p_s) m_{ts} && \text{since } \alpha_t(\alpha_{t^{-1}}(a) p_s) \in D_t D_{ts} \\ &\in D_{ts} m_{ts} \subset A \times_\alpha G. \quad \square \end{aligned}$$

McClanahan [9] constructs a regular covariant representation (π^r, u^r), of (A, G, α). We give a description which is more convenient for our purposes. For $s \in G$ let $\bar{\alpha}_s \colon A \to M(D_s)$ be the canonical homomorphism extending $\alpha_s \colon D_{s^{-1}} \to D_s$; also let χ_s be the characteristic function of $\{s\}$, and λ be the left regular representation of G. Let π be a faithful and nondegenerate representation of A on $\mathcal{H}$. Then (π^r, u^r) acts on $\mathcal{H} \otimes l^2(G)$, and is determined by

$$\pi^r \times u^r(a m_s) = \sum_t \bar{\alpha}_{t^{-1}}(a) \otimes \chi_t \lambda_s \quad \text{for} \quad a \in D_s,$$

where the sum converges in the strong* topology. The *reduced crossed product* of (A, G, α) is $A \times_{\alpha,r} G = C^*(\pi^r, u^r)$.

The following result characterises the reduced crossed product as the canonical image of $A \times_\alpha G$ in the multipliers of the double crossed product $(A \times_\alpha G) \times_{\hat{\alpha}} G$:

Proposition 3.6. *Let α be a partial action of a discrete group G on a C^*-algebra A, and let $j_{A\times_\alpha G}$ be the canonical embedding of $A \times_\alpha G$ in the crossed product by the dual coaction. Then there is an isomorphism $\theta \colon j_{A\times_\alpha G}(A \times_\alpha G) \to A \times_{\alpha,r} G$ such that*

$$\theta \circ j_{A\times_\alpha G} = \pi^r \times u^r.$$

Proof. Let M be the representation of $c_0(G)$ by multiplication operators on $l^2(G)$. By [15, Lemma 2.2], the following calculation shows $(\pi^r \times u^r, 1 \otimes M)$ is a covariant representation of $(A \times_\alpha G, G, \hat{\alpha})$: for $a \in D_s$

$$\begin{aligned}(\pi^r \times u^r)(am_s)(1 \otimes \chi_t) &= \sum_r \bar{\alpha}_{r^{-1}}(a) \otimes \chi_r \lambda_s \chi_t \\ &= \sum_r \bar{\alpha}_{r^{-1}}(a) \otimes \chi_r \chi_{st} \lambda_s \\ &= (1 \otimes \chi_{st})(\pi^r \times u^r)(am_s).\end{aligned}$$

Since $(\pi^r \times u^r)|_{(A \times_\alpha G)^{\hat{\alpha}}} = (\pi^r \times u^r)|_A = \pi^r$ is faithful, [15, Proposition 2.18] shows $\ker(\pi^r \times u^r) = \ker j_{A \times_\alpha G}$, and the result follows. □

Remark 3.7. The above proposition shows $A \times_{\alpha,r} G$ is independent up to isomorphism of the choice of faithful representation of A, so [9, Proposition 3.4] is a corollary. By [13, Proposition 2.8 (i)] and [16, Proposition 2.6 and Theorem 4.1 (2)] $\ker j_{A \times_\alpha G} = \ker(\iota \otimes \lambda) \circ \hat{\alpha}$. Thus we also obtain alternative proofs of [9, Lemma 4.1] and the half of [9, Proposition 4.2] stating that if G is amenable then $A \times_\alpha G = A \times_{\alpha,r} G$.

A coaction δ of G on a C^*-algebra B is called *normal* [14] if j_B (or equivalently, $(\iota \otimes \lambda) \circ \delta$) is faithful. The coaction $\mathrm{Ad}(j_G \otimes \iota)(w_G)$ on $j_B(B)$ is always normal, and has the same covariant representations and crossed product as δ [14, Proposition 2.6]; it is called the *normalization* of δ, and denoted δ^n. The previous proposition allows us to view the normalization $\hat{\alpha}^n$ of the dual coaction $\hat{\alpha}$ as a coaction on the reduced crossed product $A \times_{\alpha,r} G$.

Corollary 3.8. *Let (π, u) be a covariant representation of (A, G, α). Then $\ker(\pi \times u) = \ker(\pi^r \times u^r)$ if and only if π is faithful and there is a normal coaction δ of G on $C^*(\pi, u)$ with $\delta \circ (\pi \times u) = ((\pi \times u) \otimes \iota) \circ \hat{\alpha}$.*

Proof. This is immediate from the Proposition and [15, Corollary 2.19]. □

4. Landstad duality

Let δ be a coaction of the discrete group G on a C^*-algebra B. For $s \in G$ let $B_s := \{b \in B \mid \delta(b) = b \otimes s\}$ be the spectral subspace, and let $D_s = B_s B_s^* := \overline{\mathrm{sp}}\{bc^* : b, c \in B_s\}$. Then D_s is an ideal of $D_e = B_e$, and $D_{s^{-1}} = B_s^* B_s$. Let p_s denote the identity of D_s^{**} regarded as a projection in B^{**}. B_s is a $D_s - D_{s^{-1}}$ imprimitivity bimodule with inner

products ${}_{D_s}\langle x,y\rangle = xy^*$ and $\langle x,y\rangle_{D_{s^{-1}}} = x^*y$. By the discussion at the end of Section 2, the multiplier bimodule can be identified as

$$M(B_s) = \{b \in p_s B^{**} p_{s^{-1}} \mid D_s b \cup b D_{s^{-1}} \subset B_s\}.$$

Fortunately, when $s = e$ this coincides with the usual multiplier algebra $M(B_e)$ of B_e.

The following result is Landstad duality for partial actions. Condition (4.2) below was motivated by [5, Proposition 4.16].

Theorem 4.1. *Let δ be a normal coaction of a discrete group G on a C^*-algebra B. The following are equivalent*:

(i) *there is a partial action α of G on a C^*-algebra A such that (B,δ) is isomorphic to $(A \times_{\alpha,r} G, \hat{\alpha}^n)$;*

(ii) *there is a partial representation m of G in B^{**} such that*

$$m_s \in M(B_s) \quad \textit{and} \quad m_s m_s^* = p_s \quad \textit{for} \quad s \in G; \tag{4.1}$$

(iii) *there is a collection $\psi_s\colon B_s \to D_s$ of left Hilbert D_s-module isomorphisms such that*

$$\psi_{st}(xy) = \psi_s(x\psi_t(y)) \quad \textit{for} \quad x \in B_s, y \in B_t. \tag{4.2}$$

Proof. The construction in Section 3 shows that (i) implies (ii). We next show that (ii) implies (i). Since m is a partial representation, we have $m_s^* m_s = p_{s^{-1}}$. By Lemma 2.1, there are isomorphisms $\alpha_s := \operatorname{Ad} m_s\colon D_{s^{-1}} \to D_s$; we claim that α is a partial action of G on B_e. Clearly $D_e = B_e$. We must show

$$\alpha_s(D_{s^{-1}} D_t) \subset D_{st}$$

and

$$\alpha_s\alpha_t = \alpha_{st} \quad \text{on} \quad D_{t^{-1}} D_{t^{-1}s^{-1}}.$$

For the first,

$$\alpha_s(D_{s^{-1}}D_t) = m_s D_{s^{-1}} D_t D_{s^{-1}} m_s^* \subset B_s B_t B_t^* B_s^* \subset B_{st} B_{st}^* = D_{st}.$$

For the second, since m is a partial representation, (4.1) and Lemma 1.8 imply $am_s m_t = am_{st}$ for all $a \in D_s D_{st}$, or equivalently $m_{st}a = m_s m_t a$ for all $a \in D_{t^{-1}} D_{t^{-1}s^{-1}}$. Thus for such a we have

$$\alpha_{st}(a) = m_{st} a m_{st}^* = m_s m_t a m_t^* m_s^* = \alpha_s\alpha_t(a),$$

as claimed.

The pair (ι, m) is a covariant representation of (B_e, G, α), and we have $C^*(\iota, m) = B$ because $B = \overline{\sum_s B_s}$. For $a \in D_s$ we have

$$\begin{aligned} \delta \circ (\iota \times m)(am_s) = \delta(am_s) = am_s \otimes s = (\iota \times m) \otimes \iota(am_s \otimes s) \\ = ((\iota \times m) \otimes \iota) \circ \hat{\alpha}(am_s); \end{aligned}$$

since ι is faithful, (i) now follows from Corollary 3.8

Now we show that (ii) implies (iii). By Lemma 2.1, $\psi_s(x) = xm_s^*$ defines a left Hilbert module isomorphism $\psi_s\colon B_s \to D_s$. Let $x \in B_s$ and $y \in B_t$. Then there exist $a \in D_{s^{-1}}$ and $b \in D_t$ with $x = m_s a$ and $y = bm_t$. We compute:

$$\begin{aligned}\psi_{st}(xy) &= m_s abm_t m_{st}^* = m_s abm_s^* m_s m_t m_{st}^* \qquad \text{since } ab \in D_{s^{-1}}\\ &= m_s abm_s^* m_s m_t m_t^* m_s^* = m_s abm_t m_t^* m_s^*\\ &= xym_t^* m_s^* = \psi_s(x\psi_t(y)),\end{aligned}$$

giving (4.2).

Finally, to see that (iii) implies (ii), Lemma 2.1 gives $m_s \in M(B_s)$ such that

$$m_s m_s^* = p_s \quad \text{and} \quad m_s^* m_s = p_{s^{-1}}$$

and it remains to verify (1.4). But (4.2) implies

$$am_s bm_t m_{st}^* = am_s bm_t m_t^* m_s^* \quad \text{for} \quad a \in D_s, b \in D_t,$$

and letting a and b run separately through bounded approximate identities for D_s and D_t gives $m_s m_t m_{st}^* = m_s m_t m_t^* m_s^*$. □

Landstad [7, Theorem 3] originally characterised reduced crossed products by ordinary actions of a locally compact group. When the group is discrete, Landstad's characterisation is the special case of the above theorem in which $p_s = 1$ for all $s \in G$: δ is equivariantly isomorphic to the dual coaction on a reduced crossed product by an ordinary action of G if and only if there is a homomorphism $s \mapsto m_s$ of G into $UM(B)$ such that $\delta(m_s) = m_s \otimes s$ for all $s \in G$.

5. Partial actions of $\mathbb{F}_n$

McClanahan [9, Example 2.4] and (for the case $n = 1$) Exel [5] show that certain partial actions of the free group $\mathbb{F}_n$ can be reconstructed from the generators. We shall show that, for such partial actions of $\mathbb{F}_n$, our Landstad duality (Theorem 4.1) can be recast in terms of the spectral subspaces of the generators. This will give both a generalisation and a simplification of Exel's characterisation [5, Theorem 4.21] of crossed products by certain partial actions of $\mathbb{Z} = \mathbb{F}_1$.

Throughout these last two sections we shall denote by $g_1, \dots, g_n$ a fixed set of generators for the free group $\mathbb{F}_n$. A word in $\mathbb{F}_n$ is *reduced* if it is the identity or a product $s_1 s_2 \cdots s_k$ in which each s_i is either g_j or g_j^{-1} for some j, and no cancellation is possible. When we say $s_1 s_2 \cdots s_k$ is a reduced word it is implicit that each s_i has the form $g_j^{\pm 1}$.

Definition 5.1. A partial action α of $\mathbb{F}_n$ is *multiplicative* if for every reduced word $s_1 \cdots s_k$, we have $\alpha_{s_1 \cdots s_k} = \alpha_{s_1} \cdots \alpha_{s_k}$.

Thus for any multiplicative partial action we have $\alpha_{st} = \alpha_s \alpha_t$ whenever s, t are words for which there is no cancellation possible in st. The key issue is that the domains of definition must coincide, and we shall see in the next Lemma that it is relatively easy to decide whether this happens.

Lemma 5.2. *A partial action α of $\mathbb{F}_n$ is multiplicative if and only if $D_{s_1 s_2 \cdots s_k} \subset D_{s_1}$ for every reduced word $s_1 \cdots s_k$.*

Proof. The "only if" direction is clear. For the other direction, it suffices to show that $\alpha_{s_1 \cdots s_k} = \alpha_{s_1} \alpha_{s_2 \cdots s_k}$. Write $s = s_1$, $t = s_2 \cdots s_k$. Then because α_{st} is an injective map which extends $\alpha_s \alpha_t$, it is enough to show they have the same range. But

$$\begin{aligned}
\operatorname{range} \alpha_{st} &= D_{st} = D_s D_{st} \quad \text{by hypothesis} \\
&= \alpha_s(D_s^{-1} D_t) \quad \text{by Lemma 1.2} \\
&= \alpha_s \alpha_t (D_{t^{-1} s^{-1}} D_{t^{-1}}) \quad \text{by Lemma 1.2} \\
&= \operatorname{range} \alpha_s \alpha_t,
\end{aligned}$$

as required. □

Lemma 5.3. *If α is a multiplicative partial action of $\mathbb{F}_n$ on a C^*-algebra A, then the partial representation m of $\mathbb{F}_n$ in $(A \times_\alpha \mathbb{F}_n)^{**}$ satisfies $m_{s_1 \cdots s_k} = m_{s_1} \cdots m_{s_k}$ for every reduced word $s_1 \cdots s_k$.*

Proof. Again write $s = s_1$, $t = s_2 \cdots s_k$, and it is enough to show that the partial isometries m_{st} and $m_s m_t$ have the same range projection. But since $D_{st} = D_s D_{st} = \alpha_s(D_{s^{-1}} D_t)$, we have

$$p_{st} = \alpha_s(p_{s^{-1}} p_t) = m_s p_{s^{-1}} p_t m_s^* = m_s m_t m_t^* m_s^*. \quad \square$$

In the above situation, $D_{s_1 \cdots s_k} = \alpha_{s_1}(D_{s_1^{-1}} D_{s_2 \cdots s_k})$, so α is a free product of n partial actions of $\mathbb{Z}$ in the sense of McClanahan [9, Example 2.3]. In fact, a partial action of $\mathbb{F}_n$ is multiplicative if and only if it is the free product of n multiplicative partial actions of $\mathbb{Z}$. This brings up a minor inconsistency between the partial actions of Exel and McClanahan's partial actions of $\mathbb{Z}$: a partial action of $\mathbb{Z}$ in McClanahan's sense [9] is a partial action in Exel's sense [5] if and only if it is multiplicative. Examples of nonmultiplicative partial actions of $\mathbb{Z}$ are easy to come by:

Example 5.4. Suppose β is an (ordinary) action of $\mathbb{Z}$ on A. Define ideals $\{D_n\}_{n\in\mathbb{Z}}$ of A by

$$D_n = \begin{cases} A & \text{if } n \text{ is even,} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

Then define $\alpha_n = \beta_n | D_{-n}$. To see that α is a partial action, the only nontrivial condition is $\alpha_n(D_{-n}D_k) \subset D_{n+k}$. This is trivially satisfied when $n+k$ is even, and if $n+k$ is odd then n or k is odd, and $\alpha_n(D_{-n}D_k) = \{0\}$. This partial action is not multiplicative since $D_2 = A \not\subset \{0\} = D_1$.

It is easy to check whether a partial action of $\mathbb{Z}$ is multiplicative.

Lemma 5.5. *A partial action α of $\mathbb{Z}$ is multiplicative if and only if $D_n \subset D_1$ for all $n > 0$.*

Proof. It suffices to show that if $D_n \subset D_1$ for all $n > 0$ then $D_{-n} \subset D_{-1}$ for all $n > 0$. This is proven inductively by the following computation:

$$\begin{aligned} D_{-n} &= \alpha_{-n}(D_n) = \alpha_{-n}(D_1 D_n) \\ &= \alpha_{-n}\alpha_1(D_{-1}D_{n-1}) = \alpha_{1-n}(D_{-1}D_{n-1}) \\ &= \alpha_{1-n}(D_{n-1}D_{-1}) = D_{1-n}D_{-n} \subset D_{1-n}. \qquad \square \end{aligned}$$

Theorem 5.6. *Let δ be a normal coaction of $\mathbb{F}_n$ on a C^*-algebra B. Then there is an multiplicative partial action α of $\mathbb{F}_n$ on a C^*-algebra A such that (B, δ) is isomorphic to $(A \times_{\alpha,r} \mathbb{F}_n, \hat{\alpha}^n)$ if and only if*

(i) *B is generated by $B_e \cup B_{g_1} \cup \cdots \cup B_{g_n}$;*
(ii) *for each $s = g_1, \ldots, g_n$ there exists $m_s \in M(B_s)$ such that*

$$m_s m_s^* = p_s \quad \textit{and} \quad m_s^* m_s = p_{s^{-1}}.$$

Moreover, (ii) *can be replaced by*

(iii) *for each $s = g_1, \ldots, g_n$ there is a left Hilbert D_s-module isomorphism $\psi_s \colon B_s \to D_s$.*

Proof. First of all, Lemma 2.1 tells us that (ii) is equivalent to (iii). If α is a multiplicative partial action of $\mathbb{F}_n$ on A and $B = A \times_{\alpha,r} \mathbb{F}_n$, we know (ii) holds. To see (i) it suffices to show that if $s_1 \cdots s_k$ is a reduced word then

$$B_{s_1\cdots s_k} = B_{s_1} \cdots B_{s_k}, \tag{5.1}$$

and by induction it suffices to show

$$B_{s_1\cdots s_k} = B_{s_1} B_{s_2\cdots s_k}.$$

Letting $s = s_1$, $t = s_2 \cdots s_k$, and using Lemma 5.2, we have

$$\begin{aligned} B_{st} &= D_{st} m_{st} = D_s D_{st} m_{st} = \alpha_s(D_{s^{-1}} D_t) m_{st} \\ &= \alpha_s(\alpha_{s^{-1}}(D_s) D_t) m_{st} = D_s m_s D_t m_t = B_s B_t, \end{aligned}$$

as desired.

Conversely, assume that δ is a normal coaction of $\mathbb{F}_n$ on B satisfying (i) and (ii). We first show that if $s_1 \cdots s_k$ is a reduced word then (5.1) holds again. Every $x \in B_{s_1 \cdots s_k}$ is approximated by a sum of terms of the form $x_1 \cdots x_j$, with $x_i \in B_{t_i}$ and $t_i \in \{e, g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$. By applying the canonical projection of B onto $B_{s_1 \cdots s_k}$ to this sum, we see that we can assume that each $x_1 \cdots x_j \in B_{s_1 \cdots s_k}$. This forces $t_1 \cdots t_j = s_1 \cdots s_k$. Since the latter product is reduced, we can insert parentheses in the former product so that the ith chunk of t's multiplies out to s_i. Hence,

$$x_1 \cdots x_j \in B_{s_1} \cdots B_{s_k},$$

verifying (5.1).

To apply Theorem 4.1, we need a partial representation m of $\mathbb{F}_n$ in B^{**} such that for each $s \in \mathbb{F}$

$$m_s \in M(B_s) \quad \text{and} \quad m_s m_s^* = p_s. \tag{5.2}$$

We are given m_s satisfying (5.2) for $s = g_1, \dots, g_n$. Define

$$\begin{gathered} m_e = 1, \qquad m_{g_i^{-1}} = m_{g_i}^*, \qquad \text{and} \\ m_{s_1 \cdots s_k} = m_{s_1} \cdots m_{s_k} \quad \text{for any reduced word} \quad s_1 \cdots s_k. \end{gathered}$$

For $s = e, g_1^{-1}, \dots, g_n^{-1}$ it is clear that (5.2) holds. Note also that $m_s^* = m_{s^{-1}}$ for all $s \in \mathbb{F}_n$. To see that (5.2) remains true for all $s \in \mathbb{F}_n$, by induction it suffices to show that if it holds for $s = r, t$ and there is no cancellation in rt, then

$$m_{rt} \in M(B_{rt}); \tag{5.3}$$

$$m_{rt} m_{rt}^* = p_{rt}. \tag{5.4}$$

For (5.3) we need

$$m_{rt} \in p_{rt} B^{**} p_{t^{-1} r^{-1}}; \tag{5.5}$$

$$D_{rt} m_{rt} \cup m_{rt} D_{t^{-1} r^{-1}} \subset B_{rt}. \tag{5.6}$$

For (5.5) choose nets $\{x_i\}$ in B_r and $\{y_j\}$ in B_t converging to m_r and m_t strictly in $M(B_r)$ and $M(B_t)$, respectively, hence weak* in B^{**}. Then

$$m_{rt} = m_r m_t = \text{weak*} \lim_i \lim_j x_i y_j$$

and $x_i y_j \in B_r B_t \subset B_{rt} \subset p_{rt} B^{**} p_{t^{-1}r^{-1}}$, so (5.5) holds. For (5.6), recall from Lemma 2.1 that (5.2) implies $B_r^* m_r = D_{r^{-1}}$ and $D_r = B_r m_r$. Since we have already verified that (5.1) applies, we deduce that

$$\begin{aligned} D_{rt} m_{rt} &= B_{rt} B_{rt}^* m_r m_t = B_r B_t B_t^* B_r^* m_r m_t = B_r D_t D_{r^{-1}} m_t \\ &= B_r D_{r^{-1}} D_t m_t = B_r B_t = B_{rt}, \end{aligned}$$

and similarly for $m_{rt} D_{t^{-1}r^{-1}}$.

For (5.4), since m_r and m_t are partial isometries with commuting domain and range projections, $m_{rt} = m_r m_t$ is a partial isometry. Since $m_{rt} \in M(B_{rt})$, we have $m_{rt} m_{rt}^* \leq p_{rt}$. For the opposite inequality, it suffices to show $D_{rt} m_r m_t m_t^* m_r^* = D_{rt}$, and again we use (5.1):

$$\begin{aligned} D_{rt} m_r m_t m_t^* m_r^* &= B_{rt} B_{rt}^* m_r p_t m_r^* = B_r B_t B_t^* B_r^* m_r p_t m_r^* \\ &= B_r D_t D_{r^{-1}} p_t m_r^* = B_r D_{r^{-1}} D_t p_t m_r^* \\ &= B_r D_{r^{-1}} D_t m_r^* = B_r D_t D_{r^{-1}} m_r^* \\ &= B_r B_t B_t^* B_r^* = D_{rt}. \end{aligned}$$

Thus (5.4) holds, and we have proved (5.2) for all $s \in \mathbb{F}_n$.

We still need to verify that m is a partial representation. (1.2) and (1.3) are obvious, so it remains to show (1.4) for all $s, t \in \mathbb{F}_n$. Let $s = s_1 \cdots s_k$ and $t = t_1 \cdots t_j$ be the reduced spellings. Then the reduced spelling of st is of the form

$$st = s_1 \cdots s_i t_{k-i+1} \cdots t_j \quad \text{for some} \quad i \leq k.$$

Let

$$u = s_1 \cdots s_i, \quad v = s_{i+1} \cdots s_k, \quad \text{and} \quad w = t_{k-i+1} \cdots t_j.$$

Then $s = uv$, $t = v^{-1}w$, and $st = uw$, with no cancellations, so

$$\begin{aligned} m_s m_t m_t^* m_s^* &= m_{uv} m_{v^{-1}w} m_{v^{-1}w}^* m_{uv}^* = m_u m_v m_v^* m_w m_w^* m_v m_v^* m_u^* \\ &= m_u m_v m_v^* m_w m_w^* m_u^* = m_{uv} m_{v^{-1}w} m_{uw}^* = m_s m_t m_{st}^*. \quad \square \end{aligned}$$

Remark 5.7. When $n = 1$, the above theorem includes [5, Theorem 4.21], and our proof in this case is simpler than his since we can use the multiplier bimodule to go straight to the partial isometries m_s.

6. Applications and examples

(a) Cuntz algebras. The *Toeplitz-Cuntz algebra* $\mathcal{TO}_n$ is the universal C^*-algebra generated by n isometries s_i such that $\sum_i s_i s_i^*$ is a proper projection; Cuntz showed that any n isometries S_i on Hilbert space generate a faithful representation of $\mathcal{TO}_n$ if $\sum_i S_i S_i^* < 1$ [2]. The *Cuntz algebra* $\mathcal{O}_n$ is similarly generated by any family $\{S_i\}$ of isometries

satisfying $\sum_i S_i S_i^* = 1$ [2]. If g_i are generators of $\mathbb{F}_n$, then $s_i \otimes g_i \in \mathcal{TO}_n \otimes C^*(\mathbb{F}_n)$ is also a Toeplitz-Cuntz family of isometries, and hence there is a faithful, unital homomorphism $\delta\colon \mathcal{TO}_n \to \mathcal{TO}_n \otimes C^*(\mathbb{F}_n)$ such that $\delta(s_i) = s_i \otimes g_i$. Since

$$(i \otimes \delta_{\mathbb{F}_n}) \circ \delta(s_i) = i \otimes \delta_{\mathbb{F}_n}(s_i \otimes g_i) = s_i \otimes g_i \otimes g_i = (\delta \otimes i) \circ \delta(s_i),$$

δ is a coaction of $\mathbb{F}_n$ on $\mathcal{TO}_n$. Since $\{s_i \otimes \lambda_{g_i}\}$ is a Toeplitz-Cuntz family, $(i \otimes \lambda) \circ \delta$ is faithful, and δ is normal. There is a similar coaction on $\mathcal{O}_n$. We intend to apply Theorem 5.6 to these coactions.

We first recall some standard notation and facts about the Toeplitz-Cuntz family $\{s_i\}$. If $\mu = (\mu_1, \mu_2, \cdots, \mu_{|\mu|})$ is a multi-index, we write s_μ for the isometry $s_\mu = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{|\mu|}}$ in $\mathcal{TO}_n$, and g_μ for the word $g_{\mu_1} g_{\mu_2} \cdots g_{\mu_{|\mu|}}$ in $\mathbb{F}_n$, so that $\delta(s_\mu) = s_\mu \otimes g_\mu$. If we realise $\mathcal{TO}_n$ on Hilbert space, the isometries s_i have orthogonal ranges, and hence satisfy $s_i^* s_j = 0$ for $i \neq j$; it follows that every non-zero word in the s_i and s_j^* collapses to one of the form $s_\mu s_\nu^*$, for which we have $\delta(s_\mu s_\nu^*) = s_\mu s_\nu^* \otimes g_\mu g_\nu^{-1}$. A product $(s_\mu s_\nu^*)(s_\alpha s_\beta^*)$ is non-zero if and only if the multi-indices ν and α agree as far as possible; in particular, we have

$$(s_\mu s_\mu^*)(s_\nu s_\nu^*) = \begin{cases} s_\nu s_\nu^* & \text{if } \nu = (\mu_1, \cdots, \mu_{|\mu|}, \nu_{|\mu|+1}, \cdots, \nu_{|\nu|}); \\ 0 & \text{if } \nu_i \neq \mu_i \text{ for some } i \leq \min(|\mu|, |\nu|); \\ s_\mu s_\mu^* & \text{if } \mu = (\nu_1, \cdots, \nu_{|\nu|}, \mu_{|\nu|+1}, \cdots, \mu_{|\mu|}). \end{cases}$$

It follows that $D = \overline{\operatorname{sp}}\{s_\mu s_\mu^*\}$ is a commutative C^*-subalgebra of $\mathcal{TO}_n$, which is called the *diagonal subalgebra*. By convention, we write $s_\emptyset = 1$, so that D contains the identity of $\mathcal{TO}_n$.

Corollary 6.1. *There is a multiplicative partial action α of $\mathbb{F}_n$ on the diagonal subalgebra D such that $(\mathcal{TO}_n, \delta)$ is isomorphic to $(D \times_\alpha \mathbb{F}_n, \hat{\alpha})$. Similarly, $\mathcal{O}_n$ is isomorphic to the partial crossed product of its diagonal subalgebra by a multiplicative partial action of $\mathbb{F}_n$.*

Proof. Let $B = \mathcal{TO}_n$. Since the words $s_\mu s_\nu^*$ span a dense subspace of B, and the projections onto the spectral subspaces B_s are continuous, the equation $\delta(s_\mu s_\nu^*) = s_\mu s_\nu^* \otimes g_\mu g_\nu^{-1}$ implies that for each $s \in \mathbb{F}_n$,

$$B_s = \overline{\operatorname{sp}}\{s_\mu s_\nu^* : g_\mu g_\nu^{-1} = s\}.$$

In particular, $B^\delta = B_e = D$, and

$$B_{g_i} = \overline{\operatorname{sp}}\{s_i s_\mu s_\mu^*\} = s_i D \cong D = B_{g_i}^* B_{g_i} = D_{g_i^{-1}},$$

as Hilbert D-modules. Since the isometries s_i themselves generate B, it follows from Theorem 5.6 that there is a multiplicative partial action α on D such that $(B, \delta) \cong (D \times_{\alpha,r} \mathbb{F}_n, \hat{\alpha}^n)$.

To see that $D \times_\alpha \mathbb{F}_n = D \times_{\alpha,r} \mathbb{F}_n$ in this case, note that since D is generated by $\{p_s\}_{s\in\mathbb{F}_n}$, $D \times_\alpha \mathbb{F}_n$ is generated by $\{m_s\}_{s\in\mathbb{F}_n}$. Since the partial action α is multiplicative, it follows from Lemma 5.2 that $D \times_\alpha \mathbb{F}_n$ is actually generated by the Toeplitz-Cuntz family $\{m_{g_1}, \dots, m_{g_n}\}$, and by Cuntz's Theorem is therefore isomorphic to $\mathcal{TO}_n$. Thus any representation $\pi\times v$ of $D\times_\alpha\mathbb{F}_n$ in which $\sum v_i v_i^* \neq 1$ is faithful, including the regular representation whose image is $D \times_{\alpha,r} \mathbb{F}_n$. The proof for $\mathcal{O}_n$ is similar. □

Much the same arguments show that the Cuntz-Krieger algebras are partial crossed products; to avoid repetition, we shall merely realize them as reduced crossed products.

Corollary 6.2. *If A is a $\{0,1\}$-matrix satisfying condition (I) of* [3], *then the Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to a partial crossed product $D \times_{\alpha,r} \mathbb{F}_n$.*

Proof. Let $n = |A|$, and let $\{s_i\}$ be a family of partial isometries generating $\mathcal{O}_A$ and satisfying the Cuntz-Krieger relations

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*.$$

The universal property of $\mathcal{O}_A$ implies that the map $s_i \mapsto s_i \otimes g_i$ extends to a coaction $\delta\colon \mathcal{O}_A \to \mathcal{O}_A \otimes \mathbb{F}_n$ with spectral subspaces

$$B_{g_i} = \overline{\mathrm{sp}}\{s_i s_\mu s_\mu^* : A(i,\mu_1) = A(\mu_k, \mu_{k+1})\},$$

and $D := B_e = \overline{\mathrm{sp}}\{s_\mu s_\mu^*\}$. Since $s_i = \sum_j s_i s_j s_j^*$ is in B_{g_i}, the spectral subspaces generate $\mathcal{O}_A$. The ideals $D_{g_i^{-1}} := B_{g_i}^* B_{g_i}$ are given by

$$D_{g_i^{-1}} = \overline{\mathrm{sp}}\{s_\mu s_\mu^* : A(i,\mu_1) = A(\mu_k, \mu_{k+1})\},$$

and $s_i s_\mu s_\mu^* \mapsto s_i^* s_i s_\mu s_\mu^* = s_\mu s_\mu^*$ is a right Hilbert $D_{g_i^{-1}}$-module isomorphism of B_{g_i} onto $D_{g_i^{-1}}$. Thus the Corollary follows from Theorem 5.6. □

Example 6.3. We now give some related examples of systems which are not dual to partial crossed products. First of all, we claim that $\mathcal{O}_n$ is not a crossed product by a partial action of $\mathbb{Z}$ in such a way that the gauge action α of $\mathbb{T}$ agrees with the dual action. The spectral subspaces B_n for the gauge action are given by

$$B_n = \overline{\mathrm{sp}}\{s_\mu s_\nu^* : |\mu| - |\nu| = n\},$$

and all the ideals $D_n := B_n B_n^*$ are equal to the AF-core B^α of $\mathcal{O}_n$. If $(\mathcal{O}_n, \alpha)$ were the dual system of a partial crossed product $D \times_{\alpha,r} \mathbb{Z}$,

then the spectral subspaces would be isomorphic to B^α as Hilbert B^α-modules; but

$$B_1 = \overline{\mathrm{sp}}\{s_\mu s_\nu^* : |\mu| - |\nu| = 1\} = \overline{\mathrm{sp}}\{s_i t : t \in B^\alpha\}$$

is mapped isomorphically to the Hilbert B^α-module $(B^\alpha)^n$ via the map $r \mapsto (s_1^* r, \dots, s_n^* r)$, which has inverse $(t_1, \dots, t_n) \mapsto \sum s_i t_i$. That $\mathcal{O}_n$ is not a partial crossed product by $\mathbb{Z}$ underlines that stabilisation is an essential ingredient in the comment at the top of [5, page 4] to the effect that the crossed products by endomorphisms studied by Paschke [11] fit the mould of [5]. (Cuntz's description of $\mathcal{O}_n$ as a crossed product of the UHF-core A by an endomorphism does not obviously fit the pattern because the range of the endomorphism is not an ideal in the simple algebra A (see [1, Example 3.1]).

More generally, the coaction of $\mathbb{F}_n$ on $\mathcal{O}_n$ induces a coaction of any quotient G of $\mathbb{F}_n$, but arguments like those in the previous paragraph show that these are not typically the dual coaction on some decomposition $\mathcal{O}_n \cong A \times_{\alpha,r} G$ as a partial crossed product. For example, if $q\colon \mathbb{F}_3 \to \mathbb{F}_2 = \langle b_1, b_2\rangle$ is the quotient map which identifies the first two generators (say $q(g_1) = q(g_2) = b_1$, $q(g_3) = b_2$), then the composition $(i \otimes q) \circ \delta : \mathcal{O}_3 \to \mathcal{O}_3 \otimes C^*(\mathbb{F}_2)$ has the Hilbert B_e-module $B_{b_1} = \overline{\mathrm{sp}}\{s_i t : t \in B_e, i = 1, 2\}$ isomorphic to B_e^2 rather than B_e.

While it is not an immediate application of our earlier results, it is interesting to note that similar ideas give a complete characterisation of the Cuntz algebras in terms of the canonical coaction:

Proposition 6.4. *Suppose that B is a C^*-algebra with identity, carrying a coaction $\delta\colon B \to B \otimes C^*(\mathbb{F}_n)$ of $\mathbb{F}_n = \langle g_1, \cdots, g_n\rangle$. Assume that:*

(i) *the spectral subspaces B_{g_i} are isomorphic to B^δ as right Hilbert B^δ-modules;*
(ii) *$B_{g_i}^* B_{g_j} = 0$ for $i \neq j$;*
(iii) *the spaces B_{g_i} generate B as a C^*-algebra.*

Then for every word μ in the semigroup generated by $\{g_i\}$, $B_\mu B_\mu^*$ is an ideal in B^δ; as a C^*-algebra, each $B_\mu B_\mu^*$ has an identity which is a projection p_μ in B^δ. (We understand $p_\emptyset$ to be the identity of B^δ.) Assume further that:

(iv) *$B^\delta = \overline{sp}\{p_\mu : \mu$ is a word in the $g_i\}$.*

Then B is isomorphic to $\mathcal{TO}_n$ or $\mathcal{O}_n$.

Proof. We begin by observing that the first two assumptions give a copy of $\mathcal{O}_n$ or $\mathcal{TO}_n$ inside B. If $\psi_i\colon B^\delta \to B_{g_i}$ is the isomorphism

guaranteed by (i), then $s_i = \psi_i(1)$ satisfies

$$s_i^* s_i = \langle s_i, s_i \rangle_{B^\delta} = \langle \psi_i(1), \psi_i(1) \rangle_{B^\delta} = \langle 1, 1 \rangle_{B^\delta} = 1,$$

and hence is an isometry. Since $\psi_i(b) = \psi_i(1b) = s_i b$, we have $B_{g_i} = s_i B^\delta$. Thus (ii) forces $s_i^* s_j = 0$, and $\{s_i\}$ is a Toeplitz-Cuntz family.

We next claim that $B_\mu = B_{\mu_1} \cdots B_{\mu_{|\mu|}} = s_\mu B^\delta$ for all words μ in $\{g_i\}$. For any $s, t \in \mathbb{F}_n$, we have $B_s B_t \subset B_{st}$, so the problem is to show that $B_{\mu\nu} \subset B_\mu B_\nu$ for all words μ, ν in $\{g_i\}$. Note that $B_\nu^* B_\nu = B^\delta$, because $s_\nu^* s_\nu \in B_\nu^* B_\nu$ and $B_\nu^* B_\nu$ is an ideal. Thus

$$B_{\mu\nu} = B_{\mu\nu} B_\nu^* B_\nu = B_{\mu\nu} B_{\nu^{-1}} B_\nu \subset B_{\mu\nu\nu^{-1}} B_\nu = B_\mu B_\nu,$$

establishing the first equality. For the second, note that we certainly have $s_\mu B^\delta \subset B_\mu$. To see that $s_\mu B^\delta$ is all of B^δ, note that for any i, $B^\delta s_i$ is contained in the spectral subspace B_{g_i}, which we know is $s_i B^\delta$. Thus, from the first equality, we have

$$\begin{aligned} B_\mu = B_{\mu_1} \cdots B_{\mu_{|\mu|}} &= s_{\mu_1} B^\delta s_{\mu_2} B^\delta \cdots s_{\mu_{|\mu|}} B^\delta \\ &\subset s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{|\mu|}} B^\delta = s_\mu B^\delta, \end{aligned}$$

justifying the claim.

It follows from the claim that $B_\mu B_\mu^* = s_\mu B^\delta s_\mu^*$, which has identity $s_\mu s_\mu^*$, and hence (iv) says precisely that $B^\delta = \overline{\mathrm{sp}}\{s_\mu s_\mu^*\}$. Thus (iii) implies that the isometries s_i generate B, and B is either $\mathcal{TO}_n$ or $\mathcal{O}_n$ depending on whether $\sum s_i s_i^* < 1$ or $\sum s_i s_i^* = 1$. $\square$

(b) Wiener-Hopf C^*-algebras. We now consider the quasi-lattice ordered groups (G, P) of Nica [10]. Thus P is a subsemigroup of a discrete group G such that $P \cap P^{-1} = \{e\}$, and the (right) order on on G defined by $s \le t \iff s^{-1}t \in P$ has the following property: if $s_1, s_2, \ldots, s_n$ have a common upper bound in P, they also have a least upper bound $s_1 \vee s_2 \vee \cdots \vee s_n$ in P. The individual elements of G which have upper bounds in P are precisely those in $PP^{-1} = \{pq^{-1} : p, q \in P\}$, and we follow Nica in writing $\sigma(s)$ for the least upper bound in P of $s \in PP^{-1}$, and $\tau(s)$ for $s^{-1}\sigma(s)$. In general, there are many ways of writing a given element of PP^{-1} ($pq^{-1} = pr(qr)^{-1}$ for any r), and one should think of $s = \sigma(s)\tau(s)^{-1}$ as the most efficient. We refer to [10, Section 2] for the basic properties and examples.

The *Wiener-Hopf C^*-algebra* of a quasi-lattice ordered group is the C^*-algebra $\mathcal{W}(G, P)$ of operators on $l^2(P)$ generated by the isometries $\{W_p : p \in P\}$, where

$$(W_p \xi)(q) = \begin{cases} \xi(p^{-1}q) & \text{if } p^{-1}q \in P, \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that the family $\{W_pW_q^* : p,q \in P\}$ spans a dense subspace of $\mathcal{W}(G,P)$ [10, Proposition 3.2]. The *diagonal subalgebra* is $\mathcal{D} = \overline{\operatorname{sp}}\{W_pW_p^*\}$ (see [10, Section 3]).

Proposition 6.5. *There is a normal coaction δ of G on $\mathcal{W}(G,P)$ such that*

$$\delta(W_pW_q^*) = W_pW_q^* \otimes pq^{-1} \quad \textit{for} \quad p,q \in P. \tag{6.1}$$

Proof. By [14, Theorem 4.7], it suffices to show there is a reduced coaction δ^r of G on $\mathcal{W}(G,P)$ such that

$$\delta^r(W_pW_q^*) = W_pW_q^* \otimes \lambda_{pq^{-1}} \quad \text{for} \quad p,q \in P. \tag{6.2}$$

Since $\mathcal{W}(G,P)$ is by definition a subalgebra of $B(l^2(P))$, the minimal tensor product $\mathcal{W}(G,P) \otimes C_r^*(G)$ by definition acts on $l^2(P) \otimes l^2(G) = l^2(P \times G)$. We define an operator W_P on $l^2(P \times G)$ by $(W_P\xi)(p,s) = \xi(p,p^{-1}s)$; W_P is unitary with $W_P^* = W_P^{-1}$ given by $(W_P^*\xi)(p) = \xi(p,ps)$. An easy calculation shows that

$$\begin{aligned}\big(W_P(W_p \otimes 1)W_P^*\xi\big)(q,s) &= \begin{cases} \xi(p^{-1}q, p^{-1}s) & \text{if } p^{-1}q \in P, \\ 0 & \text{otherwise} \end{cases} \\ &= (W_p \otimes \lambda_p)(\xi)(q,s). \end{aligned} \tag{6.3}$$

Since the elements $W_pW_q^*$ span a dense subspace of $\mathcal{W}(G,P)$, the isometric map $T \mapsto W_P(T \otimes 1)W_P^*$ extends to a unital homomorphism $\delta^r : \mathcal{W}(G,P) \to B(l^2(P \times G))$ with range in $\mathcal{W}(G,P) \otimes C_r^*(G)$, and (6.3) implies (6.2). The coaction identity $(\delta^r \otimes i) \circ \delta^r = (i \otimes \delta_G^r) \circ \delta^r$ follows easily from (6.2). □

Theorem 6.6. *Let $\mathcal{W}(G,P) = C^*(W_p : p \in P)$ be the Wiener-Hopf C^*-algebra of a quasi-lattice ordered group. Then there is a partial action α of G on the diagonal subalgebra $\mathcal{D}$ such that the cosystem $(\mathcal{W}(G,P), G, \delta)$ of Proposition* 6.5 *is isomorphic to $(\mathcal{D} \times_{\alpha,r} G, G, \hat{\alpha}^n)$.*

Proof. Let B denote $\mathcal{W}(G,P)$. We aim to apply Theorem 4.1, so we need a partial representation m of G in B^{**} satisfying (4.1). For this, we need to identify the spectral subspaces B_s.

Lemma 6.7. *The spectral subspaces of δ are given by*

$$B_s = \begin{cases} W_{\sigma(s)}\mathcal{D}W_{\tau(s)}^* & \textit{if } s \in PP^{-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, the fixed-point algebra $B^\delta = B_e$ is the diagonal subalgebra $\mathcal{D}$.

Proof. Because $\mathcal{W}(G,P) = \overline{\mathrm{sp}}\{W_pW_q^*\}$, (6.1) and the continuity of the projection $\delta_s = (i \otimes \chi_s) \circ \delta$ onto B_s imply

$$B_s = \begin{cases} \overline{\mathrm{sp}}\{W_pW_q^* : pq^{-1} = s\} & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, B_s is precisely the subspace $\mathcal{D}_s$ described in [10, Section 3.4], so the lemma follows from [10, Section 3.5]. □

For $s \in G$ define

$$m_s = \begin{cases} W_{\sigma(s)}W_{\tau(s)}^* & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The above lemma tells us that the ideals $D_s = B_sB_s^*$ of $\mathcal{D}$ are given by

$$D_s = \begin{cases} W_{\sigma(s)}\mathcal{D}W_{\sigma(s)}^* & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$\begin{aligned} m_sm_s^* &= \begin{cases} W_{\sigma(s)}W_{\sigma(s)}^* & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise} \end{cases} \\ &= p_s. \end{aligned}$$

In particular, (1.2) holds. Since

$$\begin{aligned} m_{s^{-1}} &= \begin{cases} W_{\sigma(s^{-1})}W_{\tau(s^{-1})}^* & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} W_{\tau(s)}W_{\sigma(s)}^* & \text{if } s \in PP^{-1}, \\ 0 & \text{otherwise} \end{cases} \\ &= m_s^*, \end{aligned}$$

(1.3) holds as well. Furthermore, the above formulas imply (4.1). It remains to verify (1.4): for $s, t \in G$ we must show $m_sm_t \preceq m_{st}$. We may assume $s, t, s^{-1}t \in PP^{-1}$, since $m_sm_t = 0$ otherwise. Then

$$\begin{aligned} m_sm_t &= W_{\sigma(s)}W_{\tau(s)}^*W_{\sigma(t)}W_{\tau(t)}^* \\ &= W_{\sigma(s)\tau(s)^{-1}(\tau(s)\vee\sigma(t))}W_{\tau(t)\sigma(t)^{-1}(\sigma(t)\vee\tau(s))}^*, \end{aligned}$$

by [10, Equation(5)], while

$$m_{st} = W_{\sigma(st)}W_{\tau(st)}^*.$$

Since $W_pW_q^* \preceq W_uW_v^*$ whenever $p, q, u, v \in P$, $pq^{-1} = uv^{-1}$, and $u \leq p$, the inequality follows.

Thus Theorem 6.6 follows from Theorem 4.1. □

Remark 6.8. Nica also associates to each (G,P) a universal C^*-algebra $C^*(G,P)$ whose representations are given by representations V of P as isometries satisfying the covariance condition

$$V_pV_p^*V_qV_q^* = \begin{cases} V_{p\vee q}V_{p\vee q}^* & \text{if } p\vee q \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

If $V\colon P \to C^*(G,P)$ is the universal such representation, the map $s \mapsto V_s \otimes s$ is also covariant, and hence there is a coaction $\delta\colon C^*(G,P) \to C^*(G,P) \otimes C^*(G)$ such that $\delta(V_s) = V_s \otimes s$. It is not obvious that this coaction will be normal, and its normalisation could coact on a proper quotient of $C^*(G,P)$. However, the theory of [10] suggests that in many cases of interest the Wiener-Hopf representation induces an isomorphism of $C^*(G,P)$ onto $\mathcal{W}(G,P)$; a general theorem along these lines is proved in [8], from which Cuntz's Theorem [2] and other related results follow.

References

[1] S. Boyd, N. Keswani, and I. Raeburn, *Faithful representations of crossed products by endomorphisms*, Proc. Amer. Math. Soc. **118** (1993), 427–436.

[2] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[3] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

[4] S. Echterhoff and I. Raeburn, *Multipliers of imprimitivity bimodules and Morita equivalence of crossed products*, Math. Scand. **76** (1995), 289–309.

[5] R. Exel, *Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence*, J. Funct. Anal. **122** (1994), 361–401.

[6] ———, *Twisted partial actions a classification of stable C^*-algebraic bundles*, (preprint).

[7] M. B. Landstad, *Duality theory for covariant systems*, Trans. Amer. Math. Soc. **248** (1979), 223–267.

[8] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. (to appear).

[9] K. McClanahan, *K-theory for partial crossed products by discrete groups*, J. Funct. Anal. **130** (1995), 77–117.

[10] A. Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory **27** (1991), 1–37.

[11] W.L. Paschke, *The crossed product of a C^*-algebra by an endomorphism*, Proc. Amer. Math. Soc. **80** (1980), 113–118.

[12] G. K. Pedersen, C^*-algebras and their automorphism groups, Academic Press, London, 1979.

[13] J. Quigg, *Full C^*-crossed product duality*, J. Austral. Math. Soc. Ser. A **50** (1991), 34–52.

[14] ———, *Full and reduced C^*-coactions*, Math. Proc. Cambridge Philos. Soc. **116** (1994), 435–450.

[15] ———, *Discrete C^*-coactions and C^*-algebraic bundles*, J. Austral. Math. Soc. Ser. A (to appear).

[16] I. Raeburn, *On crossed products by coactions and their representation theory*, Proc. London Math. Soc. (3) **64** (1992), 625–652.

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

E-mail address: quigg@math.la.asu.edu

Department of Mathematics, University of Newcastle, Newcastle, New South Wales 2308, Australia

E-mail address: iain@frey.newcastle.edu.au